

Vertex-primitive groups and graphs of order twice the product of two distinct odd primes

Greg Gamble and Cheryl E. Praeger

(Communicated by A. A. Ivanov)

Abstract. A *non-Cayley number* is an integer n for which there exists a vertex-transitive graph on n vertices which is not a Cayley graph. In this paper, we complete the determination of the non-Cayley numbers of the form $2pq$, where p, q are distinct odd primes. Earlier work of Miller and the second author had dealt with all such numbers corresponding to vertex-transitive graphs admitting an imprimitive subgroup of automorphisms. This paper deals with the primitive case. First the primitive permutation groups of degree $2pq$ are classified. This depends on the finite simple group classification. Then each of these groups G is examined to determine whether there are any non-Cayley graphs which admit G as a vertex-primitive subgroup of automorphisms, and admit no imprimitive subgroups. The outcome is that $2pq$ is a non-Cayley number, where $2 < q < p$ and q and p are primes, if and only if one of $p \equiv 1 \pmod{4}$, or $q \equiv 1 \pmod{4}$, or $p \equiv 1 \pmod{q}$, or $p = 4q - 1$, or $(p, q) = (11, 7)$ or $(19, 7)$ holds.

1 Introduction

In 1983, Marušič [18] asked for a determination of the set \mathcal{NC} of natural numbers n for which there exists a vertex-transitive graph of order n , that is, on n vertices, which is not a Cayley graph. The elements of \mathcal{NC} are called *non-Cayley numbers*. Since the vertex-disjoint union of k copies of a non-Cayley, vertex-transitive graph of order n is a non-Cayley, vertex-transitive graph of order kn , it follows that any multiple of a non-Cayley number is also a non-Cayley number. Thus non-Cayley numbers with few prime divisors are of particular interest. The question of membership of \mathcal{NC} has been settled for all natural numbers which are not square-free [22, 23], or are the product of two distinct primes [20]. The results of this paper contribute to the classification, completed in [9], of the non-Cayley numbers which are products of at most three primes. The problem of membership of \mathcal{NC} for numbers of the form $2pq$, where p and q are distinct odd primes, was studied in depth by Miller and the second author in [24], and the problem was solved except for the case of graphs for which every vertex-transitive subgroup of automorphisms is vertex-primitive. The purpose of this paper is to complete the determination of non-Cayley numbers of the form $2pq$. The strategy we use for this is to determine, first, all primitive permutation

groups of degree $2pq$. Then each of these groups G is examined to determine whether there are any graphs with the above property which admit G as a vertex-primitive subgroup of automorphisms, and admit no imprimitive subgroups of automorphisms. Thus our results are of two types. First we have Theorem 1, the classification of primitive permutation groups of degree $2pq$. This result depends on the finite simple group classification.

Remark 1 (on notation). In all cases G will represent a primitive permutation group with socle T acting on a set Ω . Thus in particular T is transitive on Ω . Let $\alpha \in \Omega$. Then the action of G on Ω is permutationally isomorphic to the action of G by right multiplication on the set $[G : G_\alpha]$ of right cosets of the stabilizer G_α in G . In our proofs it will sometimes be convenient to identify Ω with $[G : G_\alpha]$. Moreover, since also T is transitive on Ω , when considering the action of T on Ω we may similarly identify Ω with $[T : T_\alpha]$. In the tables summarizing the examples we will often specify T and $\Omega = [T : T_\alpha]$. We will generally refer to *candidates* or *possibilities* that satisfy our conditions as *examples*. Often a possibility is discounted because the number of prime divisors of the degree (counted according to multiplicity) of its associated action is less than, or more than three; for brevity, we will say, in such a case, that the degree has *insufficient*, or *excess* divisors, respectively. The letter r always denotes a prime power (usually of r_0 , but sometimes we require only that r_0 divides r), and s always denotes a prime. For r a power of a prime r_0 , the r -part of an integer N , written $|N|_r$, is the highest power of r_0 that divides N (so that $|N|_r = |N|_{r_0}$). A group G is said to be *almost simple* if $T \leq G \leq \text{Aut } T$ for some non-abelian simple group T . Often we will refer to *Aschbacher classes of subgroups of a classical simple group* T ; these are the classes $\mathcal{C}_1, \dots, \mathcal{C}_8$ defined in [12]. (Note that Kleidman and Liebeck choose to include the Aschbacher class \mathcal{C}'_1 [2, Section 13] in their definition of \mathcal{C}_1 ; see [12, pp. 4, 58].) Also defined in [12] is the set \mathcal{S} (which we find convenient to label \mathcal{C}_9), a collection of quasi-simple subgroups of T . (A group is called *quasi-simple* if it is perfect and if $G/Z(G)$ is a non-abelian simple group.) Notation for the classical simple groups follows [12]; we explain this and additional notation we use, including the term *type*, after Proposition 1 in Section 2. Finally, K_m denotes the *complete graph* on m vertices.

Theorem 1. *Let p and q be distinct odd primes, where $q < p$, and suppose that G is a primitive permutation group on a set Ω of $2pq$ points. Then G is an almost simple group with a non-abelian simple socle T such that one of the following holds:*

- (i) G is not a classical group and T, p, q are given in one of the lines of Table 1;
or
- (ii) G is a classical group and T_α is an Aschbacher class \mathcal{C}_1 subgroup, (i.e. apart from line 1 of Table 2, G_α is the stabilizer of a totally singular subspace, or a non-singular subspace) and T, p, q are given in one of the lines of Table 2;
or
- (iii) $T = \text{PSL}_2(p)$, T_α is not an Aschbacher class \mathcal{C}_1 subgroup, and p, q are given in one of the lines of Table 3.

Table 1.

Socles T of non-classical almost simple groups with primitive actions on $\Omega = [T : T_x]$ of degree $2pq$, where $2 < q < p$ and p, q are primes.

T	p	q	T_x or action	Comments
A_{2pq}	p	q	natural	
A_p	p	$\frac{p-1}{4}$	$\Omega = \binom{\Sigma}{2}, \Sigma = p$	$p \geq 13$
A_{p+1}	p	$\frac{p+1}{4}$	$\Omega = \binom{\Sigma}{2}, \Sigma = p+1$	$p \geq 11$
A_{13}	13	11	$\Omega = \binom{\Sigma}{3}, \Sigma = 13$	
M_{11}	11	3	S_5	$M_{11} < M_{12} < A_{12}$ on pairs
M_{12}			$M_{10} : 2$ (2 copies)	
M_{23}	23	11	A_8	
J_1	19	7	$\text{PSL}_2(11)$	

Table 2.

Socles T of classical almost simple groups with primitive actions on $\Omega = [T : T_x]$ of degree $2pq$, where $2 < q < p$, p and q are primes, and T_x is an Aschbacher class \mathcal{C}_1 subgroup of T .

T	p	q	T_x type	Comments
$\text{PSL}_3(5)$	31	3	$P_{1,2}$	G contains a graph automorphism
$\text{PSL}_5(r)$	$\frac{r^5-1}{r-1}$	$\frac{r^2+1}{2}$	P_2	$r \equiv -1 \pmod{10}$, r prime, $r \geq 29$
$\text{PSL}_4(r)$	$\frac{r^3-1}{r-1}$	$\frac{r^2+1}{2}$	P_2	$r = 3$; or $r = 5$; or $r \equiv 11, 29 \pmod{30}$ and r prime, $r \geq 59$
$\text{PSL}_2(r)$	p	$\frac{r+1}{2p}$	P_1	$r \equiv 1 \pmod{4}$, r prime, $r \geq 29$
$\text{PSU}_3(r)$	$\frac{r^3+1}{r+1}$	$\frac{r+1}{2}$	P_1	$r = s^{2^l}$, $l \geq 0$, s prime, $s \equiv \begin{cases} 1 \pmod{12} & \text{if } l = 0 \\ 1, 49 \pmod{60} & \text{if } l > 0 \end{cases}$
$\text{P}\Omega_{2m}^+(r)$	$\frac{r^m-1}{r-1}$	$\frac{r^{m-1}+1}{2}$	P_1	$m = 2^{2^l} + 1$, $l \geq 1$, r and m prime, $r \geq 3$
$\text{P}\Omega_{2m}^-(r)$	$\frac{r^m+1}{2}$	$\frac{r^{m-1}-1}{r-1}$	P_1	$m = 2^{2^l}$, $l \geq 1$, r and $m-1$ prime, $r \geq 3$

Table 3.

Primes p, q for which $T = \text{PSL}_2(p)$ is the socle of a primitive group on $\Omega = [T : T_\alpha]$ of degree $2pq$, where $2 < q < p$, p and q are primes, and $T_\alpha < T$ is of Aschbacher class \mathcal{C}_2 – \mathcal{C}_9 .

p	q	T_α	class	Comments
p	$\frac{p+1}{4}$	D_{p-1}	\mathcal{C}_2	$T < A_{p+1}$ on pairs, $p \geq 11$
p	$\frac{p-1}{4}$	D_{p+1}	\mathcal{C}_3	$T < A_p$ on pairs, $p \geq 13$
47	23	S_4	\mathcal{C}_6	
17	3			
41	7	A_5	\mathcal{C}_9	

The fact that G is almost simple follows from the O’Nan–Scott Theorem (see, for example, [14]), using no more than the arithmetic properties of $|\Omega| = 2pq$. If $2q < p$, then the classification follows from work of Liebeck and Saxl [17]. Also, in the case where G is not a finite classical group, the classification was completed by the first author in his PhD thesis [8], which addressed a much broader research question, namely the question of determining all finite primitive permutation groups which contain an element of prime order p having at most p^2 cycles of length p . (This latter classification was completed in [8] except for the groups of affine type and the classical almost simple groups.) These ‘non-classical’ examples will be derived from [8] in Section 2, and after this we shall proceed to analyse the case of the classical almost simple groups.

Now every vertex-transitive graph Γ with G a vertex-transitive subgroup of automorphisms is a generalized orbital graph for the permutation group induced by G on the vertex set Ω . A *generalized orbital graph* for a transitive permutation group G on a set Ω is defined as follows. The group G induces a natural action on $\Omega \times \Omega$ by $(u, v)^g := (u^g, v^g)$ for all $u, v \in \Omega, g \in G$. A generalized orbital Σ is a non-empty G -invariant subset of $\Omega \times \Omega$, and the generalized orbital digraph corresponding to Σ is defined as the digraph with vertex set Ω and arcs the ordered pairs in Σ . This digraph admits G as a vertex-transitive group of automorphisms. We obtain a simple undirected graph by choosing Σ to be self-paired (in the sense that $(u, v) \in \Sigma$ if and only if $(v, u) \in \Sigma$); we avoid the diagonal (that is $(v, v) \notin \Sigma$ for all v), and we identify each pair of arcs (u, v) and (v, u) with the undirected edge $\{u, v\}$.

Thus the vertex-transitive graphs of order $2pq$ such that every vertex-transitive subgroup of automorphisms is vertex-primitive all arise as generalized orbital graphs for primitive permutation groups of degree $2pq$, that is the groups involved must be among those classified in Theorem 1. The possible values of p and q for such groups are classified in Theorem 2 below. Its proof will be given in Section 3.

Table 4.

Primes p, q for which there exists a graph Γ on $2pq$ vertices such that $\text{Aut } \Gamma$ and all vertex-transitive subgroups of $\text{Aut } \Gamma$ are vertex-primitive.

p	q	Action of $\text{Aut } \Gamma$
p	$\frac{p+1}{4}$	S_{p+1} on pairs
13	11	S_{13} on triples
23 19	11 7	M_{23} on cosets of A_8 J_1 on cosets of $\text{PSL}_2(11)$
47 17	23 3	$\text{PSL}_2(p)$ on cosets of S_4
41	7	
31	3	$\text{PSL}_2(41)$ on cosets of A_5
$\frac{r^3-1}{r-1}$	$\frac{r^2+1}{2}$	$\text{PSL}_3(5)$ on cosets of $P_{1,2}$
$\frac{r^5-1}{r-1}$	$\frac{r^2+1}{2}$	$\text{PSL}_4(r)$ on cosets of P_2 $\text{PSL}_5(r)$ on cosets of P_2
$\frac{r^m-1}{r-1}$	$\frac{r^{m-1}+1}{2}$	$P\Omega_{2m}^+(r)$ on cosets of P_1 , m an odd prime
$\frac{r^m+1}{2}$	$\frac{r^{m-1}-1}{r-1}$	$P\Omega_{2m}^-(r)$ on cosets of P_1 , $m-1$ an odd prime

Theorem 2. *Let p and q be distinct odd primes with $q < p$. Suppose that Γ is a graph on $2pq$ vertices such that $\text{Aut } \Gamma$ is vertex-primitive, and moreover every vertex-transitive subgroup of $\text{Aut } \Gamma$ is vertex-primitive. Then p, q are as in Table 4. Conversely there are examples of such graphs corresponding to each line of Table 4.*

Theorem 2 provides a characterization of integers $2pq$ such that a vertex-transitive graph Γ of order $2pq$ exists for which all vertex-transitive subgroups of automorphisms are vertex-primitive. We remark that such a characterization is not strictly necessary for the determination of the non-Cayley numbers of the form $2pq$, since to supplement the results of [24] we only require knowledge of such numbers for which there exists a vertex-primitive non-Cayley graph of order $2pq$. However, Theorem 2 is of some interest in its own right. To our knowledge, the first construction of a vertex-transitive graph such that all vertex-transitive subgroups of automorphisms are vertex-primitive was given in [27]. Theorem 2 gives several infinite families of such graphs.

Finally we draw together the results of this paper and those of [24] to complete the determination of necessary and sufficient conditions for $2pq$ to belong to \mathcal{NC} .

Theorem 3. *Let p and q be odd primes with $q < p$.*

- (a) *Some proper divisor of $2pq$ lies in \mathcal{NC} (and hence also $2pq \in \mathcal{NC}$) if and only if one of the following holds:*
- (i) $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$;
 - (ii) $p \equiv 1 \pmod{q^2}$;
 - (iii) $p = 11, q = 7$.
- (b) *$2pq \in \mathcal{NC}$ but no proper divisor of $2pq$ lies in \mathcal{NC} if and only if $p \equiv q \equiv 3 \pmod{4}$, and one of the following holds:*
- (i) $p \equiv 1 \pmod{q}$ and $p \not\equiv 1 \pmod{q^2}$;
 - (ii) $q = \frac{1}{4}(p + 1)$;
 - (iii) $p = 19, q = 7$.

2 Primitive groups of degree $2pq$

Let G, T, Ω be as in Theorem 1, and suppose first that T is not a classical simple group.

Proposition 1. *If T is not a classical simple group, then T, p, q are as in one of the lines of Table 1.*

Proof. Firstly, we have $T = A_{2pq}$ under the natural action. Otherwise, the examples in Table 1 follow from the results summarized in [8, Table 3.3] and detailed in [8, Chapter 6]. Note that the entries of [8, Table 3.3] are listed in order of $\log_p m$ (where, in our case, $m = 2q$); so, only the possibilities up to the line with entry 1.428 in the column headed $\log_p m$ (since, in our case p is at least 5, and $\log_p(2q) < \log_p(2p) = 1 + \log_p 2$) and the families of possibilities with entry '> 1' in the $\log_p m$ column, need to be considered. Most of the non-classical possibilities listed in [8, Table 3.3] are easily discounted on trivial grounds, for example, because the degree of the action is odd or not square-free or has excess divisors. We only give the arguments for those candidates that are excluded for somewhat non-trivial reasons. Lines 2 and 3 of Table 1, A_p and A_{p+1} , are established as follows: [8, Table 3.3] lists $T = A_c$ acting of degree $\binom{c}{2}$ on unordered pairs from a set of size c . Thus $p|c$ or $p|c - 1$, whence $p \in \{c, c - 1\}$, since $|\Omega| = 2pq$ with p the larger odd prime divisor. In a similar manner, in line 4 we see that the only example with $T = A_c$ acting on triples occurs for $c = 13$. Lines 5–8 of Table 1 are straightforward to establish from [8, Table 3.3].

Now consider $G_2(r)$, acting of degree $(r^6 - 1)/(r - 1)$. Since the degree is even, r must be odd. This leads to

$$|\Omega| = 2(r^2 + r + 1) \cdot \frac{r + 1}{2} \cdot (r^2 - r + 1)$$

which has excess divisors and so does not give an example in Table 1. Similarly, the $G_2(r)$ action of degree $(r^6 - 1)(r + 1)/(r - 1)$ is discounted.

We assume for the rest of this section that T is a classical simple group of dimension $n \geq 2$ over the field \mathbb{F}_r (or over \mathbb{F}_{r^2} in the unitary case), so that T is $\text{PSL}_n(r)$, $\text{PSp}_n(r)$ (with n even, $n \geq 4$), $\text{P}\Omega_n^\varepsilon(r)$ (with either $\varepsilon = \pm$, $n \geq 8$, and n even, or $\varepsilon = \circ$, $n \geq 7$ and nr odd), or $\text{PSU}_n(r)$ (with $n \geq 3$). When we wish to suppress the parameters n, r of a classical simple group we will use the term *type*; and we use a (roman) T for variable references to a type. For example, ' $T = T_n(r)$ where T is of type PSL ' means ' $T = \text{PSL}_n(r)$ '. Note that, when r is odd, $\text{SO}_n(r)$ contains no non-trivial scalars and hence $\text{P}\Omega_n(r) \cong \Omega_n(r)$, and so we abbreviate the type $\text{P}\Omega^\circ$ to Ω° or just Ω .

Let \hat{T} denote the corresponding subgroup of $\text{GL}_n(r)$ (except for the unitary case), that is, for T of types PSL , PSp or $\text{P}\Omega^\varepsilon$, $\hat{T} = \text{SL}_n(r)$, $\text{Sp}_n(r)$ or $\Omega_n^\varepsilon(r)$ respectively, and in the unitary case, $\hat{T} = \text{SU}_n(r) < \text{GL}_n(r^2)$. Let \hat{T}_α be the preimage in \hat{T} of the stabilizer T_α , so that

$$|\hat{T} : \hat{T}_\alpha| = |T : T_\alpha| = 2pq.$$

Whenever G corresponds to a subgroup of $\text{P}\Gamma\text{L}_n(r)$, \hat{G} denotes a preimage of G containing \hat{T} . Thus corresponding to the groups G, T, T_α we have the preimages $\hat{G}, \hat{T}, \hat{T}_\alpha$, where \hat{T}_α should be read as ' T_α -hat' (rather than a point-stabilizer of \hat{T}).

We shall usually analyse the various possibilities according to the nature of the \hat{T}_α -action on the underlying vector space $V = V_n(r)$. In most cases G corresponds to a subgroup of $\text{P}\Gamma\text{L}_n(r)$. However, this is not always the case: the exceptional cases occur when

- (i) $T = \text{PSL}_n(r)$ with $n \geq 3$ if G contains an element which interchanges k -spaces and $(n - k)$ -spaces,
- (ii) $T = \text{PSp}_4(r)$ if G contains a graph automorphism, and
- (iii) $T = \text{P}\Omega_8^+(r)$ if G contains a triality automorphism.

Case (i) has been taken into account in Aschbacher's work [2] and will be dealt with in our analysis of the general case. To a lesser extent this is true for the groups $\text{PSp}_4(r)$, and not true at all for the groups $\text{P}\Omega_8^+(r)$ in case (iii). So we deal with cases (ii) and (iii) now, before proceeding further. Note that, in case (ii), we may assume that $r > 2$, since $\text{PSp}_4(2)' \cong \text{A}_6$, and this case has already been treated.

In our analysis we shall refer to *primitive prime divisors* of numbers of the form $r^i - 1$; these are primes which divide $r^i - 1$ but which do not divide $r^j - 1$ for any j satisfying $1 \leq j < i$. It was proved by Zsigmondy [32] in 1892 that, if r and i are

integers with $r \geq 2$, $i \geq 3$ and $(r, i) \neq (2, 6)$, then $r^i - 1$ has a primitive prime divisor. Also $r^2 - 1$ has a primitive prime divisor unless $r = 2^l - 1$ for some l .

Proposition 2. *If $T = \text{PSp}_4(r)$ with $r \geq 3$, then G contains no graph automorphisms.*

Proof. Let r_0 be the prime dividing r . Suppose that G contains a graph automorphism. Then the possibilities for \hat{T}_α are given in [15, p. 96] (see also [2, Section 14]). If \hat{T}_α is a parabolic subgroup then

$$|\hat{T} : \hat{T}_\alpha| = (r^2 + 1)(r + 1)^2 / \gcd(2, r - 1)$$

which is not of the form $2pq$. On the other hand, if \hat{T}_α is of type $\text{O}_2^\pm(r) \text{wr } \text{S}_2$, $\text{O}_2^-(r^2).2$, $\text{Sp}_4(r_1)$ (with $r = r_1^c$ with c prime), or $\text{Sz}(r)$ (with r an odd power of 2), then $|\hat{T} : \hat{T}_\alpha|$ is divisible by r_0^2 .

The groups G of type $\text{O}_8^+(r)$ containing a triality automorphism also need separate attention. We treat these below using the classification by Kleidman [10] of the maximal subgroups of $\text{P}\Omega_8^+(r)$.

Proposition 3. *If $T = \text{P}\Omega_8^+(r)$ with $r \geq 2$, then G contains no triality automorphisms.*

Proof. Suppose that $G_0 = T = \text{P}\Omega_8^+(r)$, so that $G_0 \leq G \leq \text{Aut } G_0$. Suppose also that M is a maximal subgroup of G . Then the Results Matrix [10, Table I] lists all the possibilities for $M_0 = G_0 \cap M$.

Let r_i denote a primitive prime divisor of $r^i - 1$. Note that r_4 exists and also r_6 exists except for $r = 2$. If $r = 2$ then, for each of the possibilities of M , $|G : M|$ is either odd or divisible by 4. So we may assume that $r > 2$. For lines 1–8 of the Results Matrix, $|G : M|$ is divisible by $r_6 r_4^2$. For lines 9–14, $M_0 \leq \Omega_7(r)$ and so $|G : M|$ is divisible by $r^3(r^4 - 1)/\gcd(4, r^4 - 1)$; and similarly, for all remaining lines of the Results Matrix, $|G : M|$ is divisible by r^2 . Thus, in all cases $|G : M|$ is not square-free.

We now proceed to the general analysis. The case in which \hat{T}_α is reducible on V gives rise to a number of examples and we consider this case first. Note that, for $1 \leq k < n$, the number of k -dimensional subspaces of V is

$$\text{sub}_{n,k}(r) := \prod_{i=1}^k \left(\frac{r^{n+1-i} - 1}{r^i - 1} \right).$$

Proposition 4. *If \hat{T}_α is reducible on V then we have the examples in Table 2.*

Proof. The cases that we must consider are summarized in [12, Table 4.1A]. Suppose first that $\hat{T} = \text{SL}_n(r)$, and that G contains a graph automorphism, so that $n \geq 3$. Then \hat{T}_α is the stabilizer of a pair of subspaces W, U of V such that $\dim W = k$ and $\dim U = n - k$ with $1 \leq k \leq \frac{1}{2}n$, and either $W \subset U$, $k \neq \frac{1}{2}n$ (case **L**: type $\text{P}_{k,n-k}$ of

[12, Table 4.1A)] or $V = W \oplus U$ (case **L**: type $\mathrm{GL}_k(r) \oplus \mathrm{GL}_{n-k}(r)$ of [12, Table 4.1A]). In the former case, $|\hat{T} : \hat{T}_\alpha| = \mathrm{sub}_{n,n-k}(r) \mathrm{sub}_{n-k,k}(r)$; while in the latter case, $|\hat{T} : \hat{T}_\alpha| = \mathrm{sub}_{n,k}(r) \cdot r^{k(n-k)}$. Since $|\hat{T} : \hat{T}_\alpha|$ is square-free, we must be in the former case, and since $|\hat{T} : \hat{T}_\alpha| \equiv 2 \pmod{4}$, r is odd and $k = 1$ (otherwise $|\Omega|$ has excess divisors or the smallest prime divisor is larger than 2). Thus

$$|\hat{T} : \hat{T}_\alpha| = \frac{r^n - 1}{r - 1} \cdot \frac{r^{n-1} - 1}{r - 1} = 2pq.$$

One of the factors $(r^j - 1)/(r - 1)$ is an odd prime, and for this factor the exponent j must be an odd prime. The other factor is twice an odd prime, and is of the form $(r^j + 1)(r^j - 1)/(r - 1)$ for some integer j ; we must have $j = 1$ and hence $n = 3$, $p = (r^3 - 1)/(r - 1)$ and $q = (r + 1)/2$. Moreover, $r \neq 3$ (otherwise $q = 2$) and $r \not\equiv 1 \pmod{3}$ (otherwise 3 is a proper divisor of p). Hence $r \equiv -1 \pmod{3}$ and so $3|q$. Thus $q = 3$, $r = 5$ and $p = 31$. This is line 1 of Table 2.

Thus we may now assume that G corresponds to a subgroup of $\mathrm{PGL}_n(r)$, and T_α is a subgroup in $\mathcal{C}_1 \setminus \mathcal{C}'_1$ (see [12, pp. 4, 58]).

Suppose first that \hat{T}_α is the stabilizer of a direct sum decomposition $W \perp W^\perp$, where W is a non-singular k -dimensional subspace (case **U**: type $\mathrm{GU}_k(r) \perp \mathrm{GU}_{n-k}(r)$, case **S**: type $\mathrm{Sp}_k(r) \perp \mathrm{Sp}_{n-k}(r)$, case **O $^\varepsilon$** : type $\mathrm{O}_k^{\varepsilon_1}(r) \perp \mathrm{O}_{n-k}^{\varepsilon_2}(r)$ where $\varepsilon = \varepsilon_1 \varepsilon_2$, or case **O $^\pm$** : type $\mathrm{Sp}_{n-2}(r)$ with $k = 1$, of [12, Table 4.1A]). Then, in each case, we find (as with case **L**: type $\mathrm{P}_{k,n-k}$ above) that $|\hat{T} : \hat{T}_\alpha|$ is divisible by r^2 and hence is not square-free.

All remaining examples in Aschbacher class \mathcal{C}_1 are of type P_k . We consider these for the various possibilities for T .

Case 1. $T = \mathrm{PSL}_n(r)$. Here T_α is the stabilizer in T of a k -subspace of degree $|T : T_\alpha| = \mathrm{sub}_{n,k}(r)$. Now $k \leq 2$ and r is odd, since otherwise $|T : T_\alpha|$ has excess divisors or is odd.

Suppose first that $k = 1$. Then

$$|T : T_\alpha| = r^{n-1} + r^{n-2} + \cdots + 1 \equiv n \pmod{2}.$$

So n is even, say $n = 2m$. Since $|T : T_\alpha| \equiv 2 \pmod{4}$, we have $r \equiv 1 \pmod{4}$ and m odd. But then

$$|T : T_\alpha| = 2 \cdot \frac{r^m - 1}{r - 1} \cdot \frac{r + 1}{2} \cdot \frac{r^m + 1}{r + 1}$$

has excess divisors unless $m = 1$, whence $n = 2$, $2pq = r + 1$ as in line 4 of Table 2. So assume that $k \neq 1$; then we have r odd, $k = 2$ and degree

$$\frac{(r^n - 1)(r^{n-1} - 1)}{(r - 1)(r^2 - 1)}.$$

Recall that $(r^n - 1)/(r - 1) \equiv n \pmod{2}$.

Subcase 1(a). n is odd. Here

$$p = (r^n - 1)/(r - 1) \quad \text{and} \quad q = \frac{1}{2}(r^{n-1} - 1)/(r^2 - 1).$$

Since q is an integer, $n \equiv 1 \pmod{4}$, whence $n = 4l + 1$, say, and so

$$q = \frac{(r^{2l} + 1)(r^{2l} - 1)}{2(r^2 - 1)}.$$

Since q is prime, $l = 1$, i.e. $n = 5$, $q = \frac{1}{2}(r^2 + 1)$, and primality of p forces r to be prime and $r \neq 3, 5$. If $r \equiv \pm 2 \pmod{5}$ then 5 is a proper divisor of q . If $r \equiv 1 \pmod{5}$ then 5 is a proper divisor of p . Thus $r \equiv -1 \pmod{10}$ and hence we have line 2 of Table 2.

Subcase 1(b). n is even. Here

$$p = (r^{n-1} - 1)/(r - 1) \quad \text{and} \quad q = \frac{1}{2}(r^n - 1)/(r^2 - 1),$$

and we find that $n = 4$, $q = \frac{1}{2}(r^2 + 1)$, $p = (r^3 - 1)/(r - 1)$ and r is prime. Further, if $r \neq 3$ and $r \equiv \pm 2 \pmod{5}$ then 5 is a proper divisor of q . Also, if $r \equiv 1 \pmod{3}$, then 3 is a proper divisor of p . Hence either $(r, p, q) = (3, 13, 5)$ or $(5, 31, 13)$, or $r \equiv 11, 29 \pmod{30}$, and so we have line 3 of Table 2.

Case 2. $T = \text{PSU}_n(r)$. Here our actions have degree

$$\frac{\prod_{i=1}^{2k} (r^{n+1-i} - (-1)^{n+1-i})}{\prod_{j=1}^k (r^{2j} - 1)}$$

where $n \geq 3$ and $1 \leq k \leq \lfloor \frac{1}{2}n \rfloor$ and we find that $n = 3$ and p, q are as listed in line 5 of Table 2.

Case 3. $T = \text{PSp}_n(r)$ or $T = \Omega_{n-1}(r)$, n even, $n \geq 4$. Here our actions have degree

$$\frac{\prod_{i=1}^{2k} (r^{n+1-i} - 1)}{\prod_{j=1}^k (r^j - 1)^2}$$

where $1 \leq k \leq \frac{1}{2}n$, and, with a similar argument to that of Case 1, we find that $k = 2$, in which case $n > 4$ gives a degree with excess divisors and $n = 4$ gives a degree that is divisible by 4. So there are no examples in this case.

Case 4. $T = \text{P}\Omega_n^+(r)$, $n = 2m \geq 8$. Firstly, if $k = m$ then our actions have degree

$$\prod_{i=1}^{k-1} \left(\frac{r^{2m-2i} - 1}{r^i - 1} \right)$$

which has excess divisors, since $k - 1 \geq 3$. Thus $k < \frac{1}{2}n = m$ and our actions have degree

$$\frac{r^m - 1}{r^{m-k} - 1} \cdot \prod_{i=1}^k \left(\frac{r^{2m-2i} - 1}{r^i - 1} \right)$$

whence, in order to avoid excess divisors, we must have $k = 1$. Thus

$$|T : T_\alpha| = \frac{r^m - 1}{r^{m-1} - 1} \cdot \frac{r^{2m-2} - 1}{r - 1} = 2 \cdot \frac{r^m - 1}{r - 1} \cdot \frac{r^{m-1} + 1}{2},$$

from which we obtain line 6 of Table 2.

Case 5. $T = \text{P}\Omega_n^-(r)$, $n = 2m \geq 8$, T_α a \mathcal{C}_1 type P_k subgroup. Here our actions have degree

$$\frac{r^m + 1}{r^{m-k} + 1} \cdot \prod_{i=1}^k \left(\frac{r^{2m-2i} - 1}{r^i - 1} \right)$$

and a very similar line of reasoning to Case 4 yields line 7 of Table 2.

Having dealt with reducible (i.e. the Aschbacher \mathcal{C}_1) subgroups of T , we now deal with the irreducible non-quasi-simple (i.e. the Aschbacher $\mathcal{C}_2, \dots, \mathcal{C}_8$) subgroups of T and the quasi-simple (which we label \mathcal{C}_9) subgroups of T . See [12] for a fuller description of the classes $\mathcal{C}_1, \dots, \mathcal{C}_8$, and the \mathcal{C}_9 ($=\mathcal{S}$, in [12]) subgroups. Propositions 5 and 6 reduce the remaining problem to the case when $n = 2$, which is dealt with in Proposition 7.

Proposition 5. *If T_α is in one of the Aschbacher classes \mathcal{C}_2 – \mathcal{C}_8 , then $T = \text{PSL}_2(r)$, for some prime-power r .*

Proof. We show that there are no example pairs (T, T_α) where T_α is a \mathcal{C}_2 – \mathcal{C}_8 subgroup of T for $n \geq 3$. The information we need is in Tables 3.5.A–F of [12] and the Theorems 4.i.j specified by columns I and II of these tables. Our main strategy is to show that $|T : T_\alpha|$ is not square-free, by showing that r^2 divides $|T : T_\alpha|$ (our usual argument for classes $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_7, \mathcal{C}_8$), or that r_0^2 divides $|T : T_\alpha|$ where $r = r_0^e$ for some prime e (particularly for class \mathcal{C}_5) or that $|T : T_\alpha|_2$ is at least 4. For class \mathcal{C}_6 , we use [13, Theorem 4.2] which bounds $|T_\alpha|$ above by r^{2n+4} ; by assuming $|T_\alpha| =$

Table 5.
 $|T : T_\alpha|_r$ for T_α a \mathcal{C}_2 subgroup of T , where $n = kl$.

T	T_α type	$ T : T_\alpha _r$	Conditions
$\text{PSL}_n(r)$	$\text{GL}_k(r) \text{wr } S_l$	$\frac{r^{n(n-k)/2}}{ l! _r}$	$n \geq 3, k \geq 1, l \geq 2$
$\text{PSU}_n(r)$	$\text{GU}_k(r) \text{wr } S_l$		
$\text{PSU}_n(r)$	$\text{GU}_{n/2}(r^2).2$	$\frac{r^{n^2/4}}{ 2 _r}$	n even
$\text{PSp}_n(r)$	$\text{Sp}_n(r) \text{wr } S_l$	$\frac{r^{(n/2)((n-k)/2)}}{ l! _r}$	k even, $l \geq 2$
	$\text{GL}_{n/2}(r^2).2$	$r^{(n/4)((n/2)+1)}$	r odd
$\Omega_n(r)$	$\text{O}_k(r) \text{wr } S_l$	$\frac{r^{((n-1)/2)^2 - l((k-1)/2)^2}}{ l! _r}$	$k, l \geq 3$
	$\text{O}_1(r) \text{wr } S_n$	$\frac{r^{((n-1)/2)^2}}{ l! _r}$	r an odd prime
$\text{P}\Omega_n^\pm(r)$	$\text{O}_k^\varepsilon(r) \text{wr } S_l$	$\frac{r^{(n/2)((n/2)-1) - (lk/2)((k/2)-1)}}{ 2^{l-1} _r l! _r}$	$\varepsilon \in \{+, -\}, k$ even, $l \geq 2$
		$\frac{r^{(n/2)((n/2)-1) - l((k-1)/2)^2}}{ l! _r}$	r odd, $\varepsilon = \circ, k$ even, $l \geq 2$
	$\text{O}_1(r) \text{wr } S_n$	$\frac{r^{(n/2)((n/2)-1)}}{ l! _r}$	r an odd prime
	$\text{O}_{n/2}(r)^2$	$r^{((n/2)-1)((n/4)+(1/2))}$	$r, n/2$ odd
$\text{P}\Omega_n^+(r)$	$\text{GL}_{n/2}(r).2$	$\frac{r^{(n/4)((n/2)-1)}}{ 2 _r}$	

$|T|/(2pq)$ we deduce a contradiction except for a few easily-treated possibilities for which n is small. Let $n \geq 3$. We now consider each class in turn.

The class \mathcal{C}_2 . The pairs (T, T_α) that we need to consider are listed in Table 5; the values $|T : T_\alpha|_r$ are readily determined from the information contained in [12]. We consider just two of the cases listed in Table 5; for the remaining cases, it is straightforward to show that $|T : T_\alpha|_r$ is at least r^2 .

Suppose that $T = \text{PSL}_n(r)$ with T_α of type $\text{GL}_k(r) \text{wr } S_l$, or $T = \text{PSU}_n(r)$ with T_α of type $\text{GU}_k(r) \text{wr } S_l$. Then $n = kl \geq 3, k \geq 1, l \geq 2$ and

Table 6.
 $|T : T_\alpha|_r$ for T_α a \mathcal{C}_3 subgroup of T , where $n = ke$ and e is prime.

T	T_α type	$ T : T_\alpha _r$	Conditions
$\text{PSL}_n(r)$	$\text{GL}_k(r^e)$	$\frac{r^{n(n-k)/2}}{ e _r}$	$n \geq 3$
$\text{PSU}_n(r)$	$\text{GU}_k(r^e)$		
$\text{PSp}_n(r)$	$\text{PSp}_k(r^e)$	$\frac{r^{(n/2)((n-k)/2)}}{ e _r}$	
	$\text{GU}_{n/2}(r)$	$r^{n^2/4}$	r odd
$\Omega_n(r)$	$\text{O}_k(r^e)$	$\frac{r^{((n-1)/2)^2 - e((k-1)/2)^2}}{ e _r}$	$n \geq 9, k > 1$
$\text{P}\Omega_n^\pm(r)$	$\text{GU}_{n/2}(r).2$	$\frac{r^{(n/4)((n/2)-1)}}{ (n/2, 2) _r}$	
	$\text{O}_k^\pm(r^e)$	$\frac{r^{(n/2)((n-k)/2)}}{ (e, 2)e _r}$	
	$\text{O}_{n/2}(r^2)$	$r^{((n/2)-1)((n/4)+(1/2))}$	$r, n/2$ odd

$$|T : T_\alpha|_r = r^{n(n-k)/2} / |l!|_r;$$

and since $|l!|_r \leq r^{(l-1)/(r-1)}$ we have $|T : T_\alpha|_r \geq r^2$.

Now consider $T = \text{PSp}_n(r)$ with T_α of type $\text{Sp}_k(r) \text{ wr } S_l$. Then $n = kl \geq 4$, k is even, $l \geq 2$ and $|T : T_\alpha|_r = r^{(n/2)((n-k)/2)} / |l!|_r$. For $n \geq 6$ we have

$$|T : T_\alpha|_r \geq r^{(n/2) \cdot (n/4) - ((n/2)-1)} > r^2.$$

Thus we may assume that $n = 4$ and $r > 2$ (since $\text{PSp}_4(2)' \cong \text{A}_6$ has already been treated). Then, for r odd, $|T : T_\alpha|_r = r^2$; and for r even, 2^2 divides r , whence

$$|T : T_\alpha|_2 \geq (2^2)^2 / |2!|_2 = 2^3.$$

The class \mathcal{C}_3 . The pairs (T, T_α) we need to consider and corresponding values of $|T : T_\alpha|_r$ are listed in Table 6. As with \mathcal{C}_2 , we take two cases to illustrate our arguments; it is straightforward to show that $|T : T_\alpha|_r \geq r^2$ for the remaining possibilities.

Suppose that $T = \text{PSL}_n(r)$ with T_α of type $\text{GL}_k(r^e)$, or $T = \text{PSU}_n(r)$ with T_α of type $\text{GU}_k(r^e)$. Then

$$|T : T_\alpha|_r = r^{n(n-k)/2} / |e|_r \geq r^{n(n-k)/2-1}$$

which is always strictly greater than r .

Table 7.
 $|T : T_\alpha|_r$ for T_α a \mathcal{C}_4 subgroup of T , where $n = kl$.

T	T_α type	$ T : T_\alpha _r$	Conditions
$\mathrm{PSL}_n(r)$	$\mathrm{GL}_k(r) \otimes \mathrm{GL}_l(r)$	$r^{(n/2)(n-1)-(k/2)(k-1)-(l/2)(l-1)}$	$2 \leq k < l$
$\mathrm{PSU}_n(r)$	$\mathrm{GU}_k(r) \otimes \mathrm{GU}_l(r)$		
$\mathrm{PSp}_n(r)$	$\mathrm{Sp}_k(r) \otimes \mathrm{O}_l(r)$	$r^{(n/2)^2-(k/2)^2-((l-1)/2)^2}$	k even, $l \geq 3$, r odd
	$\mathrm{Sp}_k(r) \otimes \mathrm{O}_l^\pm(r)$	$r^{(n/2)^2-(k/2)^2-(l/2)((l/2)-1)}$	k even, $l \geq 3$, r odd
$\Omega_n(r)$	$\mathrm{O}_k(r) \otimes \mathrm{O}_l(r)$	$r^{((n-1)/2)^2-((k-1)/2)^2-((l-1)/2)^2}$	$3 \leq k < l$
$\mathrm{P}\Omega_n^\pm(r)$	$\mathrm{Sp}_k(r) \otimes \mathrm{Sp}_l(r)$	$\frac{r^{(n/2)((n/2)-1)^2-(k/2)^2-(l/2)^2}}{ 2 _r}$	$2 \leq k < l$
	$\mathrm{O}_k^{\varepsilon_1}(r) \otimes \mathrm{O}_l^{\varepsilon_2}(r)$	$r^{(n/2)((n/2)-1)-(k/2)((k/2)-1)-(l/2)((l/2)-1)}$	$\varepsilon_i \in \{+, -\}$, $k, l \geq 4$
$\mathrm{P}\Omega_n^\pm(r)$	$\mathrm{O}_k^\pm(r) \otimes \mathrm{O}_l(r)$	$r^{(n/2)((n/2)-1)-(k/2)((k/2)-1)-((l-1)/2)^2}$	$k \geq 4, l \geq 3$

Now suppose that $T = \mathrm{PSp}_n(r)$ with T_α of type $\mathrm{PSp}_k(r^e)$. Then $n = ke \geq 4$, e is prime and

$$|T : T_\alpha|_r = r^{(n/2)((n-k)/2)} / |e|_r.$$

For $n \geq 6$ we have

$$|T : T_\alpha|_r \geq r^{(n/2) \cdot (n/4)-1} > r^3.$$

This leaves $n = 4$ (whence $k = e = 2$) and $r > 2$ to check. For r odd, $|T : T_\alpha|_r = r^2$; and for r even, 2^2 divides r , whence $|T : T_\alpha|_2 \geq (2^2)^2 / |2|_2 = 2^3$.

The class \mathcal{C}_4 . The cases to be considered here are listed in Table 7. Consider the case when $T = \mathrm{PSp}_n(r)$ with T_α of type $\mathrm{Sp}_k(r) \otimes \mathrm{O}_l^\varepsilon(r)$, where $n = kl \geq 6$, k is even and $l \geq 3$, so that $k \leq \frac{1}{3}n$ and $l \leq \frac{1}{2}n$. Thus, whether $\varepsilon = \circ$ or $\varepsilon \in \{+, -\}$, we have

$$|T : T_\alpha|_r \geq r^{(n/2)^2-((n/3)/2)^2-((n/2-1)/2)^2} > r^2.$$

The arguments for the other cases are similar.

The class \mathcal{C}_5 . The cases to be considered are listed in Table 8. Where the table lists $|T : T_\alpha|_{r_0}$ it is straightforward to show that $|T : T_\alpha|_{r_0} \geq r_0^2$; in the remaining cases $|T : T_\alpha|_r$ is listed and for these cases it is easy to show that $|T : T_\alpha|_r \geq r^2$.

The class \mathcal{C}_6 . Here [13, Theorem 4.2] bounds $|T_\alpha|$ above by r^{2n+4} . Thus, for primes p, q such that $2 < q < p$, we first assume that $|T : T_\alpha| = 2pq$, or equivalently that

Table 8.

$|T : T_\alpha|_{r_0}$ or $|T : T_\alpha|_r$ for T_α a \mathcal{C}_5 subgroup of T , where $r = r_0^e$, e prime.

T	T_α type	$ T : T_\alpha _{r_0}$ or $ T : T_\alpha _r$	Conditions
$\text{PSL}_n(r)$	$\text{GL}_n(r_0)$	$r_0^{(e-1)n(n-1)/2}$	
$\text{PSU}_n(r)$	$\text{GU}_n(r_0)$		
$\text{PSU}_n(r)$	$\text{O}_n(r)$	$r^{(n^2-1)/4}$	r, n odd
	$\text{O}_n^\pm(r)$	$r^{n^2/4}$	r odd, n even
	$\text{Sp}_n(r)$	$r^{(n/2)((n/2)-1)}$	n even
$\text{PSp}_n(r)$	$\text{PSp}_n(r_0)$	$r_0^{(e-1)(n/2)^2}$	
$\Omega_n(r)$	$\text{O}_n(r_0)$	$r_0^{(e-1)((n-1)/2)^2}$	
$\text{P}\Omega_n^\pm(r)$	$\text{O}_n^e(r_0)$	$r_0^{(e-1)(n/2)((n/2)-1)}$	$e \in \{+, -\}$

$|T_\alpha| = |T|/(2pq)$ and so derive a lower bound for T_α that contradicts [13, Theorem 4.2] except for some small possibilities for n .

Case 1. $T = \text{PSL}$ or PSU with T_α of type $s^{2k}\text{Sp}_{2k}(s)$. Here $n = s^k$, s is prime and $s \neq r_0$, where r_0 is the prime divisor of r .

For $n \geq 5$, assume for a contradiction that $|T_\alpha| = |T|/(2pq)$ for distinct primes p, q . Then pq is certainly not larger than $(r^n - \varepsilon)(r^{n-2} - \varepsilon)/(r - \varepsilon)$, where $\varepsilon = +$ if $T = \text{PSL}$ or $\varepsilon = -$ if $T = \text{PSU}$. Thus

$$|T_\alpha| = \frac{|T|}{2pq} \geq \frac{1}{2} r^{n(n-1)/2} (r^2 - 1)(r^4 - 1) \geq \frac{1}{2} r^{2n} \cdot 3 \cdot \frac{15}{16} r^4 > r^{2n+4}$$

contradicting the upper bound of [13, Theorem 4.2].

For $n = 4$ we have $T_\alpha \in \{2^4.\text{A}_6, 2^4.\text{Sp}_4(2)\}$ and r is odd, whence

$$|T : T_\alpha|_r = r^6/|6!|_r = r^6/|3^2.5|_r \geq r^4.$$

For $n = 3$ we have $T_\alpha \in \{3^2.\text{Q}_8, 3^2.\text{Sp}_2(3)\}$, 3 does not divide r , and $r \neq 2$. Hence $|T : T_\alpha|_r = r^3$ if r is odd, or $|T : T_\alpha|_2 \geq 2^6/2^3 = 2^3$ if r is even.

Case 2. $T = \text{PSp}$ with T_α of type $2_-^{1+2k}.\text{O}_{2k}^-(2)$. Here $4 \leq n = 2^k$ and r is an odd prime. Firstly $n < 8$, by a similar argument to that of Case 1, and hence $n = 4$: here $k = 2$ and hence

$$|T_\alpha| = 2^4.2^2(2^2 + 1)(2^4 - 1) = 2^6.5^2.3,$$

whence $|T : T_\alpha|_r = r^{2^2}/|5^2.3|_r \geq r^2$.

Table 9.
 $|T : T_\alpha|_r$ for T_α a \mathcal{C}_7 subgroup of T , where $n = k^l$.

T	T_α type	$ T : T_\alpha _r$	Conditions
$\text{PSL}_n(r)$	$\text{GL}_k(r) \text{wr } S_l$	$\frac{r^{n(n-1)/2 - lk(k-1)/2}}{ l! _r}$	$k \geq 3, l \geq 2$
$\text{PSU}_n(r)$	$\text{GU}_k(r) \text{wr } S_l$		
$\text{PSp}_n(r)$	$\text{Sp}_n(r) \text{wr } S_l$	$\frac{r^{(n/2)^2 - l(k/2)^2}}{ l! _r}$	$k \geq 2, l \geq 3$
$\Omega_n(r)$	$\text{O}_k(r) \text{wr } S_l$	$\frac{r^{((n-1)/2)^2 - l((k-1)/2)^2}}{ l! _r}$	$k \geq 3, l \geq 2$
$\text{P}\Omega_n^+(r)$	$\text{Sp}_k(r) \text{wr } S_l$	$\frac{r^{(n/2)((n/2)-1) - l(k/2)^2}}{ l! _r}$	$k \text{ even}, l \geq 2$
	$\text{O}_k^\pm(r) \text{wr } S_l$	$\frac{r^{(n/2)((n/2)-1) - (lk/2)((k/2)-1)}}{ l! _r}$	$k \geq 4, l \geq 2$

Case 3. $T = \text{P}\Omega^+$ with T_α of type $2_+^{1+2k}.\text{O}_{2k}^+(2)$. Here $8 \leq n = 2^k$ and r is an odd prime. Firstly $n < 10$, by a similar argument to that of Case 1. This leaves $n = 8$ to check: here $|T_\alpha|_r = |3^2.5.7|_r \leq r^2$ and hence $|T : T_\alpha|_r \geq r^{(n/2)((n/2)-1)-2} = r^{10}$.

The class \mathcal{C}_7 . For each \mathcal{C}_7 type, $n = k^l$ with at least the restriction that $l \geq 2, k \geq 2$. In particular, we have $n = k^l \geq kl$ and $l \leq n/k \leq \frac{1}{2}n$. The cases we need to consider are listed in Table 9. We consider two cases; for the remaining cases it is straightforward to show that $|T : T_\alpha|_r \geq r^2$.

Suppose that $T = \text{PSL}_n(r)$ with T_α of type $\text{GL}_k(r) \text{wr } S_l$, or that $T = \text{PSU}_n(r)$ with T_α of type $\text{GU}_k(r) \text{wr } S_l$. Then $n = k^l \geq 9, k \geq 3, l \geq 2$ and

$$|T : T_\alpha|_r = \frac{r^{n(n-1)/2 - lk(k-1)/2}}{|l!|_r} \geq r^{n(n-k)/2 - (l-1)} \geq r^{n^2/4 - n/3 + 1} > r^2.$$

Suppose that $T = \text{P}\Omega_n^+(r)$ with T_α of type $\text{Sp}_k(r) \text{wr } S_l$. Then $8 \leq n = k^l$ with k even and $l \geq 2$, and

$$|T : T_\alpha|_r = r^{(n/2)((n/2)-1) - l(k/2)^2} / |l!|_r.$$

If $n > 8$ then n is at least 16, whence $|T : T_\alpha|_r \geq r^2$. If $n = 8$ then $k = 2, l = 3$, $|T : T_\alpha|_r = r^{4.3-3} / |3!|_r \geq r^8$.

The class \mathcal{C}_8 . The cases to be considered are listed in Table 10. Suppose first that $T = \text{PSL}_n(r)$. For T_α of type $\text{U}_n(r_0)$ where the table lists $|T : T_\alpha|_{r_0}$ it is straightforward to show that $|T : T_\alpha|_{r_0} \geq r_0^3$. In the remaining $\text{PSL}_n(r)$ cases $|T : T_\alpha|_r \geq r^2$.

Table 10.
 $|T : T_\alpha|_{r_0}$ or $|T : T_\alpha|_r$ for T_α a \mathcal{C}_8 subgroup of T

T	T_α type	$ T : T_\alpha _{r_0}$ or $ T : T_\alpha _r$	Conditions
$\text{PSL}_n(r)$	$\text{Sp}_n(r)$	$r^{(n/2)((n/2)-1)}$	n even
	$\text{O}_n(r)$	$r^{(n^2-1)/4}$	r, n odd
	$\text{O}_n^\pm(r)$	$r^{n^2/4}$	r odd, n even
	$\text{U}_n(r_0)$	$r_0^{n(n-1)/2}$	$r = r_0^2$
$\text{PSp}_n(r)$	$\text{O}_n^\pm(r)$	$\frac{r^{n/2}}{ 2 _r}$	r even

Now suppose that $T = \text{PSp}_n(r)$ with T_α of type $\text{O}_n^\pm(r)$. Then r is even and $|T : T_\alpha|_r = r^{n/2}/|2|_r$. If $n \geq 6$ then $|T : T_\alpha|_r \geq r^{n/2-1} \geq r^2$. If $n = 4$ then since $r > 2$ and r is even we have $|T : T_\alpha|_2 \geq \frac{1}{2}(2^2)^{n/2} = 2^3$.

Thus we have shown that either $|T : T_\alpha|$ is not square-free or, in the case of \mathcal{C}_6 groups, $|T|/(2pq)$ is greater than the $|T_\alpha|$ lower bound provided by [13, Theorem 4.2], and hence there can be no examples for T_α a \mathcal{C}_2 – \mathcal{C}_8 subgroup of T for $n > 2$.

Proposition 6. *If T_α is a quasi-simple subgroup of T in the Aschbacher class \mathcal{C}_9 , then $T = \text{PSL}_2(r)$, for some prime-power r .*

Proof. Assume that $n > 2$. By [13, Theorem], if T_α is a quasi-simple subgroup of $T = \text{T}_n(r)$ then $|T_\alpha| < r^{3n}$ or, in some cases, T'_α may be A_c where c is $n+1$ or $n+2$.

For $T = \text{PSp}, \Omega, \text{P}\Omega^+$ or $\text{P}\Omega^-$, we must check that $T'_\alpha \neq \text{A}_c$ where c is $n+1$ or $n+2$. Since $|T|_r \geq r^e$ where $e \geq n(n-2)/4$ and $|(n+2)!|_r \leq r^d$ where

$$d \leq (n+2-1)/(r-1) \leq n+1,$$

we find that at least r^2 divides $|T : T_\alpha|$, if $n \geq 7$. If $n < 7$ then $n = 4$ or 6 and $T = \text{PSp}$, $|T|_r = r^{n^2/4}$; again, in these cases, $|T : T_\alpha| > r$ (since $T \neq \text{Sp}_4(2)'$). Thus we may assume that $T'_\alpha \neq \text{A}_c$.

Since we require $|T : T_\alpha| = |T|/|T_\alpha|$ to be of the form $2pq$, most possibilities may be dealt with by showing, for any prime divisors p, q of $|T|$, that $|T|/(2pq) \geq r^{3n}$. Indeed, this inequality holds in the following cases: $n \geq 6$ for $T = \text{PSL}$ or PSU , $n \geq 8$ for $T = \text{PSp}$, $n \geq 9$ for $T = \Omega$, or $n \geq 10$ for $T = \text{P}\Omega^+$ or $\text{P}\Omega^-$. The argument for the case $T = \text{PSL}$ is as follows (the arguments for the other cases are similar). Suppose that $T = \text{PSL}_n(r)$ and $n \geq 6$. Then, considering upper bounds for prime divisors of $|T|$, we see that $2pq$ is at most $2(r^n - 1)(r^{n-2} - 1)/(r - 1)^2$, whence

$$\begin{aligned} \frac{|T|}{2pq} &\geq \frac{1}{2} r^{n(n-1)/2} (r^2 - 1)(r^{n-3} - 1)(r^{n-1} - 1) \\ &\geq r^{3n-3} \cdot \frac{1}{2} \cdot \frac{3}{4} r^2 \cdot \frac{7}{8} r^3 \cdot \frac{31}{32} r^5 > r^{3n}. \end{aligned}$$

When $n = 5$ for $T = \text{PSL}$ or PSU , [13, Theorem 4.2] gives a better bound ($|T_\alpha| < r^{2n+4}$), and an argument similar to that above shows that $|T|/(2pq) > r^{2n+4}$.

Thus the following possibilities remain: $n \in \{3, 4\}$ for $T = \text{PSL}$ or PSU , $n \in \{4, 6\}$ for $T = \text{PSp}$, $n = 7$ for $T = \Omega$, and $n = 8$ for $T = \text{P}\Omega^+$ or $\text{P}\Omega^-$. These remaining possibilities are discounted by checking the possible \mathcal{C}_9 ($= \mathcal{S}$, in [11]) subgroups listed in Kleidman's tables [11]. In each case, r^2 divides $|T : T_\alpha|$.

Now we are left with the case when $T = \text{PSL}_2(r)$, with $T_\alpha = G_\alpha \cap T$ where G_α is a maximal irreducible subgroup of G . All such groups were classified by Dickson [6]. (Here we include the quasi-simple possibility also, namely, $T_\alpha = \text{A}_5$.) Since we have already considered the groups $T = \text{A}_n$, we may assume here that $r \geq 7$ and $r \neq 9$.

Proposition 7. *If $T = \text{PSL}_2(r)$, where $r \geq 7$ and $r \neq 9$, then $r = p$ and the examples are listed in Table 3.*

Proof. We have several cases to check. In each case where there are examples we will find that $r = p$.

Case 1. T_α is a dihedral subgroup of T of index $\frac{1}{2}r(r+1)$ (resp. $\frac{1}{2}r(r-1)$). Here we obtain the examples listed in line 1 (resp. line 2) of Table 3.

Case 2. T_α is a linear group over a subfield of \mathbb{F}_r , i.e. $T_\alpha = \text{PSL}_2(r^{1/k})$.c where $c = \gcd(k, 2, r-1)$, $r = r_0^e$ and $k|e$. Here

$$|T : T_\alpha| = \frac{r(r^2 - 1)}{cr^{1/k}(r^{2/k} - 1)}.$$

Observe that $r^{1-1/k} = r_0^{(e/k)(k-1)}$ must be prime. Hence r_0 is prime and $e/k = k-1 = 1$, i.e. $k = 2 = e$. Now r_0 cannot be 2. So $|T : T_\alpha| = \frac{1}{2}r_0(r_0^2 + 1)$ with r_0 odd, whence $r_0^2 \equiv 1 \pmod{4}$ and $\frac{1}{2}(r_0^2 + 1) \equiv 1 \pmod{2}$ which makes $|T : T_\alpha|$ odd, and hence there are no examples in this case.

Case 3. $T_\alpha = \text{A}_4$ or S_4 . Here r is an odd prime and $|T : T_\alpha| = r(r^2 - 1)/(24c)$, where c is 1 if $r \equiv \pm 3 \pmod{8}$ or c is 2 if $r \equiv \pm 1 \pmod{8}$. Hence $p = r$ and

$$q = (r^2 - 1)/(48c) = (r-1)(r+1)/(48c).$$

Since $r+1$ and $r-1$ differ by 2, one has 2-part at least $8c$ and the other 2. In particular, $r \equiv \pm 1 \pmod{8}$, so that $c = 2$. Thus, for $\varepsilon = \pm 1$, we have $(r + \varepsilon)/16 \in \{1, 3\}$, from which we obtain line 3 ($\varepsilon = 1$) and line 4 ($\varepsilon = -1$) of Table 3.

Case 4. $T_\alpha = \text{A}_5$. Here $r \equiv \pm 1 \pmod{10}$, r is prime and $|T : T_\alpha| = r(r^2 - 1)/120$. Hence $p = r$ and $q = (r^2 - 1)/240$. Suppose first that $|r+1|_2 \geq 8$ and $|r-1|_2 = 2$. Then either $r \equiv -1 \pmod{10}$, whence $(r+1)/40 \in \{1, 3\}$; or $r \equiv 1 \pmod{10}$ whence one of $\frac{1}{8}(r+1)$, $\frac{1}{10}(r-1)$ is 3. Each possibility leads to a contradiction. Hence $|r-1|_2 \geq 8$ and $|r+1|_2 = 2$. Similar checks here yield the one example listed as the final line of Table 3.

3 Non-Cayley numbers $2pq$ arising from vertex-primitive graphs

In this section we examine each of the primitive permutation groups G on a set Ω of size $2pq$ (where $2 < q < p$ and p, q are primes) occurring in Theorem 1 to decide whether or not there exists a graph Γ with vertex set Ω such that

- (i) $G \leq \text{Aut } \Gamma$, and
- (ii) every subgroup L of $\text{Aut } \Gamma$ such that L is transitive on Ω acts primitively on Ω .

First we outline the strategy we will use, and then we give the details of our arguments, which will prove Theorem 2.

Let $G \leq \text{Sym } \Omega$ be primitive on Ω of degree $2pq$ and let Γ be a graph with vertex set Ω admitting G as a subgroup of automorphisms (with its given action on Ω) which is therefore vertex-primitive. From our discussion towards the end of Section 1, we know that Γ is a generalized orbital graph for G relative to some self-paired generalized G -orbital in $\Omega \times \Omega$. If Γ were the complete graph K_{2pq} then $\text{Aut } \Gamma = S_{2pq}$ would contain transitive imprimitive subgroups (for example, there is a transitive cyclic subgroup of order $2pq$), and so (ii) above would not hold. Hence we may assume that $\Gamma \neq K_{2pq}$. In particular, G is not 2-transitive on Ω , and hence G is not one of the groups in Table 1, line 1 or in Table 2, lines 4 or 5. Our first task is to identify $A := \text{Aut } \Gamma$. Since A contains G , A is primitive of degree $2pq$ and hence is almost simple and its socle and stabilizer are listed in one of the the lines of Tables 1, 2 or 3. Often (but not always) we will find that $A = G$. Once we have determined A we will check that the socle of A is primitive on Ω (since otherwise (ii) would not hold), and then decide whether there exist any proper vertex-transitive subgroups L of A (not containing the socle of A), and if so whether such subgroups can act imprimitively on Ω . If L is such a subgroup then we have a proper factorization

$$A = LA_\alpha$$

of the almost simple group A , where neither L nor A_α contain the socle of A , and at least A_α is a maximal subgroup of A . If M is maximal such that M contains L and M does not contain $\text{soc } A$, then $A = MA_\alpha$ and either M is a maximal subgroup of A and all possibilities for (A, M, A_α) are listed in [15, Tables 1–6 and Theorem D], or else the possibilities for (A, M, A_α) are given in [16]. In some cases we will not need to resort to these tables as the fact that $2pq$ divides $|L|$ may be sufficient to identify all possibilities for L . In yet other cases, when A is sufficiently small, we are able to find the information we need in [5]. Thus we are able to determine whether A has any transitive imprimitive subgroups.

We now consider the possibilities for G , occurring in Tables 1, 2 and 3, line by line, and find all p, q (and representative groups/graphs) satisfying (i) and (ii). Let $G \leq \text{Aut } \Gamma$ where Γ has vertex set Ω , and Γ is not a complete graph.

Table 1, line 1. Since G is 2-transitive this gives no examples, as discussed above.

Table 1, line 2. The group $T = A_p$ has rank 3 with T_α -orbits of lengths 1, $2(p-2)$, $\binom{p-2}{2}$. Since Γ is not a complete graph, it follows (for example by checking Tables 1, 2 and 3) that $A = \text{Aut } \Gamma = S_p$. A Frobenius subgroup $Z_p.Z_{p-1}$ is transitive and imprimitive on Ω .

Table 1, line 3. As in the previous case, T has rank 3, and since $\Gamma \neq K_{p(p+1)/2}$, we have $A = \text{Aut } \Gamma = S_{p+1}$. Let $L < A$, with L transitive on Ω . Then $\frac{1}{2}p(p+1)$ divides $|L|$, and L is transitive on pairs from the set Σ of size $p+1$ on which A acts naturally. It follows that L is transitive on Σ , and (since p divides $|L|$) in fact L is 2-transitive on Σ . Since $p+1 = 4q$ is not a prime power, L is therefore an almost simple 2-transitive group of degree $p+1$ on Σ . (In particular, L is not a Frobenius group in its action on Σ , and hence L is not regular on Ω . Thus Γ is not a Cayley graph.) In fact (see, for example [4, p. 8]) L is $\text{PSL}_2(p)$, $\text{PGL}_2(p)$, M_{11} or M_{12} (with $p = 11$), A_{p+1} , or S_{p+1} . All these groups are primitive on Ω and so we have line 1 of Table 4.

Table 1, line 4. $T = A_{13}$, $|\Omega| = 286 = 2.11.13$. Since $\Gamma \neq K_{286}$ it follows from Theorem 1 that $A = \text{Aut } \Gamma = S_{13}$. Let $L \leq A$, with L transitive on Ω . Then 13.11 divides $|L|$ and so L is 3-transitive of degree 13, and hence L contains A_{13} . Thus we have line 2 of Table 4.

Table 1, lines 5, 6. Here $|\Omega| = 66$ and either $T = M_{11}$ has rank 4 with subdegrees (i.e. T_α -orbit lengths) 1, 15, 20, 30, or $T = M_{12}$ has rank 3 with subdegrees 1, 20, 45 (see, for example, [26, pp. 35 and 38]). Moreover (by Theorem 1, essentially) either $\text{Aut } \Gamma = S_{12}$ on pairs, or $\text{Aut } \Gamma = M_{11}$ (and Γ has valency 15, 30, 35, or 50). In each case, $\text{Aut } \Gamma \leq S_{12}$ in its action on pairs, and the argument for Table 1 line 3 shows that there are examples as in line 2 of Table 4. We note that the graphs arising when $T = M_{12}$ are the same as those for Table 1 line 3, while additional graphs occur when $T = M_{11}$. However, we do not of course obtain new values for p, q .

Table 1, line 7. Here $G = T = M_{23}$ and we deduce from Theorem 1 that $\text{Aut } \Gamma = M_{23}$. A transitive subgroup L of M_{23} has order divisible by $2.11.23$ and it follows from [5, p. 71] that $L = M_{23}$. Thus we have an example: line 3 of Table 4.

Table 1, line 8. Here $G = T = J_1$ and we deduce from Theorem 1 that $\text{Aut } \Gamma = J_1$. A transitive subgroup L of J_1 has order divisible by $2.7.19$ and it follows from [5, p. 36] that $L = J_1$, giving line 4 of Table 4.

Table 2, line 1. Here $T = \text{PSL}_3(5)$ and $G = \text{Aut } T = T.2$; by Theorem 1, $A = \text{Aut } \Gamma = T.2$. Let $L < A$, L transitive on Ω . By [5, p. 38] the only proper subgroup L with order divisible by $2.3.31$ is $L = 31 : 6$, the normalizer of a Singer cycle. We claim, however, that this group is not transitive on Ω . It is sufficient to show that an involution in L fixes a point of Ω .

From the fact that $L = 31 : 6$ we know that L contains an involution $g \in A \setminus T$. From the character table [5, p. 38], g is in class 2B and $C_T(g)$ has order 120, and

(from the table of maximal subgroups in [5, p. 38]) $C_T(g) \cong S_5$ is the stabilizer of a conic C in the projective plane $\text{PG}_2(5)$. We note that Ω is the set of pairs (U, W) of subspaces of a 3-dimensional vector space over \mathbb{F}_5 where $\dim U = 1$, $\dim W = 2$, and $U \subset W$. Thus Ω can be identified with the set of incident point-line pairs in $\text{PG}_2(5)$.

Now C is a set of 6 points, no two collinear. There are 15 secant lines to C (meeting C in two points), 6 tangent lines, and 10 external lines to C , and $C_T(g)$ leaves invariant each of these three subsets of lines. The element g interchanges points and lines of $\text{PG}_2(5)$, and so maps C to a subset C^g of 6 lines which is invariant under $C_T(g)$. It follows that C^g is the set of 6 tangent lines to C . Let $\alpha \in C$ and let l be the tangent line to C on α . Then $l = \beta^g$ for some $\beta \in C$. Since $C_T(g) = S_5$ acts 2-transitively on the 6 points of C , there is an involution $h \in C_T(g)$ which interchanges α and β . Note that $\alpha^{hg} = \beta^g = l$, and $l^{hg} = \alpha^{(hg)^2} = \alpha$, since $(hg)^2 = h^2g^2 = 1$ (since $hg = gh$). Thus hg is an involution in $A \setminus T$ and hg fixes the point (α, l) of Ω . By [5, p. 38] again, there is only one conjugacy class of involutions contained in $A \setminus T$, the class 2B, and hence hg and g are conjugate in A . Thus g also fixes a point of Ω . Thus we have proved that A satisfies (ii), giving line 8 of Table 4.

Table 2, lines 2–7. We note first that in lines 4 and 5, T is 2-transitive, so no examples arise here. Consider the other lines. Since Γ is not a complete graph, it follows from Theorem 1 and [15] that $A = \text{Aut } \Gamma$ has socle T . Suppose that $L \leq A$ and L is transitive on Ω . Then $2pq$ divides $|L|$ and we have $A = LA_\alpha$ with A_α of type P_2 for lines 2–3 or P_1 for lines 6–7. By [15, Tables 1–6] and [14] there are no possibilities for L except subgroups containing T . Thus we have lines 9–12 of Table 4.

Table 3. Here $T = \text{PSL}_2(p)$. The values for p, q in line 1 already occur in line 1 of Table 4. In line 2, $T_\alpha = D_{p+1}$ and we have $T = T_\alpha P_1$ (where P_1 is a parabolic subgroup of T). The subgroup P_1 is transitive and imprimitive on Ω , so we have no examples from this line. In the final lines 3–5 the subgroup T_α is S_4 or A_5 , and no proper subgroup of A ($=T$ or $T.2$), not containing T , is transitive on Ω . This gives lines 5–7 of Table 4.

This completes the proof of Theorem 2.

4 Proof of Theorem 3

It follows from [1, 7] that $2p \in \mathcal{NC}$ (respectively $2q \in \mathcal{NC}$) if and only if $p \equiv 1 \pmod{4}$ (respectively $q \equiv 1 \pmod{4}$). Also, necessary and sufficient conditions for pq to lie in \mathcal{NC} are given in [23, Theorem 1]. (We note that in parts (b), (c) and (d) of [23, Theorem 1] one of $p - 1, q - 1$ is divisible by 4 so that (a)(i) holds.) Thus part (a) is proved.

Suppose then that $2pq \in \mathcal{NC}$ but that no proper divisor lies in \mathcal{NC} . Then by part (a), $p \equiv q \equiv 3 \pmod{4}$. If there is a non-Cayley vertex-transitive graph of order $2pq$ which has a transitive imprimitive subgroup of automorphisms then, by [24, Theorem 2] we have $p \equiv 1 \pmod{q}$ (and, of course, $p \not\equiv 1 \pmod{q^2}$ by part (a)), or $(p, q) = (11, 3)$ which satisfies $q = \frac{1}{4}(p + 1)$. On the other hand, if there is a non-Cayley

vertex-transitive graph of order $2pq$ such that every vertex-transitive subgroup of automorphisms is vertex-primitive, then by Theorem 2, p and q are as in one of the lines of Table 4. In line 1, we have $q = \frac{1}{4}(p+1)$. Lines 2, 6, 7 do not occur, since $13 \equiv 17 \equiv 41 \equiv 1 \pmod{4}$. For lines 3, 5, 8, condition (b)(i) holds, while for line 4 condition (b)(iii) holds. This leaves the infinite families. We note that, if t is any odd integer, then $\frac{1}{2}(t^2+1) \equiv 1 \pmod{4}$. Thus in lines 9, 10, 11 we have $q \equiv 1 \pmod{4}$, while in line 12, $p \equiv 1 \pmod{4}$. Thus one of (b)(i)–(iii) holds.

Conversely, if p, q are as in (b)(i) or in (b)(ii)–(iii), then $2pq \in \mathcal{NC}$ by [24, Theorem 2] or Theorem 2 respectively.

Acknowledgment. The authors would like to thank the referees for their comments and suggestions. They also thank Tim Penttila for suggesting the geometrical argument used for Table 2, line 1, in the proof of Theorem 2.

References

- [1] B. Alspach and R. J. Sutcliffe. Vertex-transitive graphs of order $2p$. In *Second international conference on combinatorial mathematics* (New York, 1978) (New York Acad. Sci., 1979), pp. 18–27.
- [2] M. Aschbacher. On the maximal subgroups of finite classical groups. *Invent. Math.* **76** (1984), 469–514.
- [3] N. L. Biggs. *Algebraic graph theory*. Cambridge Tracts in Math. 67 (Cambridge University Press, 1974).
- [4] P. J. Cameron. Finite permutation groups and finite simple groups. *Bull. London Math. Soc.* **13** (1981), 1–22.
- [5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson. *An atlas of finite groups* (Clarendon Press, 1985).
- [6] L. E. Dickson. *Linear groups with an exposition of the Galois theory* (Dover, 1958).
- [7] R. Frucht, J. E. Graver and M. E. Watkins. The groups of the generalized Petersen graphs. *Proc. Cambridge Philos. Soc.* **70** (1971), 211–218.
- [8] G. Gamble. Elements of prime order in primitive permutation groups. Ph.D. thesis. University of Western Australia (1996). <http://maths.uwa.edu.au/~gregg/GGthesis.ps>
- [9] M. A. Iranmanesh and C. E. Praeger. On non-Cayley vertex-transitive graphs of order a product of three primes. Preprint (1999).
- [10] P. B. Kleidman. The maximal subgroups of the finite 8-dimensional orthogonal groups. $\text{P}\Omega_8^+(q)$ and of their automorphism groups *J. Algebra* **110** (1987), 173–242.
- [11] P. B. Kleidman. The subgroup structure of some finite simple groups. Ph.D. thesis. University of Cambridge (1987).
- [12] P. B. Kleidman and M. W. Liebeck. *The subgroup structure of the finite classical groups*. London Math. Soc. Lecture Note Series 129 (Cambridge University Press, 1990).
- [13] M. W. Liebeck. On the orders of maximal subgroups of the finite classical groups. *Proc. London Math. Soc.* (3) **50** (1985), 426–446.
- [14] M. W. Liebeck, C. E. Praeger and J. Saxl. On the O’Nan–Scott theorem for primitive permutation groups. *J. Austral. Math. Soc. Ser. A* **44** (1988), 389–396.
- [15] M. W. Liebeck, C. E. Praeger and J. Saxl. *The maximal factorizations of the finite simple groups and their automorphism groups*. Mem. Amer. Math. Soc. 432 (American Mathematical Society, 1990).

- [16] M. W. Liebeck, C. E. Praeger and J. Saxl. On factorizations of almost simple groups. *J. Algebra* **185** (1996), 409–419.
- [17] M. W. Liebeck and J. Saxl. Primitive permutation groups containing an element of large prime order. *J. London Math. Soc.* (2) **31** (1985), 237–249.
- [18] D. Marušič. Cayley properties of vertex symmetric graphs. *Ars Combin.* **16** (1983), 297–302.
- [19] D. Marušič, Vertex transitive graphs and digraphs of order p^k . In *Cycles in graphs* (Burnaby, B.C., 1982), North-Holland Math. Stud. 115 (North-Holland, 1985), pp. 115–128.
- [20] D. Marušič and R. Scapellato. Characterizing vertex-transitive pq -graphs with an imprimitive automorphism subgroup. *J. Graph Theory* **16** (1992), 375–387.
- [21] D. Marušič, R. Scapellato and B. Zgrablić. On quasiprimitive pqr -graphs. *Algebra Colloq.* **2** (1995), 295–314.
- [22] B. D. McKay and C. E. Praeger. Vertex-transitive graphs which are not Cayley graphs, I. *J. Austral. Math. Soc. Ser. A* **56** (1994), 53–63.
- [23] B. D. McKay and C. E. Praeger. Vertex-transitive graphs that are not Cayley graphs, II. *J. Graph Theory* **22** (1996), 321–334.
- [24] A. A. Miller and C. E. Praeger. Non-Cayley vertex-transitive graphs of order twice the product of two odd primes. *J. Algebraic Combin.* **3** (1994), 77–111.
- [25] C. E. Praeger, C. H. Li and A. C. Niemeyer. Finite transitive permutation groups and finite vertex-transitive graphs. In *Graph symmetry (Montreal, PQ, 1996)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 497 (Kluwer Acad. Publ., 1997), pp. 277–318.
- [26] C. E. Praeger and L. H. Soicher. *Low rank representations and graphs for sporadic groups* (Cambridge University Press, 1997).
- [27] C. E. Praeger and M. Y. Xu. Vertex-primitive graphs of order a product of two distinct primes. *J. Combin. Theory Ser. B* **59** (1993), 245–266.
- [28] D. J. S. Robinson. *A course in the theory of groups*. Graduate Texts in Math. 80 (Springer-Verlag, 1982).
- [29] G. F. Royle and C. E. Praeger. Constructing the vertex-transitive graphs of order 24. *J. Symbolic Comput.* **8** (1989), 309–326.
- [30] A. Seress. On vertex-transitive non-Cayley graphs of order pqr . *Discrete Math.* **182** (1998), 279–292.
- [31] H. Wielandt. *Finite permutation groups* (Academic Press, 1964).
- [32] K. Zsigmondy. Zur Theorie der Potenzreste. *Monatsh. für Math. u. Phys.* **3** (1892), 265–284.

Received 5 April, 1999; revised 21 October, 1999

G. Gamble, Centre for Discrete Mathematics & Computing, Department of Computer Science & Electrical Engineering, The University of Queensland, St. Lucia, Queensland 4072, Australia

C. E. Praeger, Department of Mathematics and Statistics, The University of Western Australia, Nedlands, Western Australia 6907, Australia